Semi-Global Leader-Following Consensus of Multiple Linear Systems with Position and Rate-Limited Actuators

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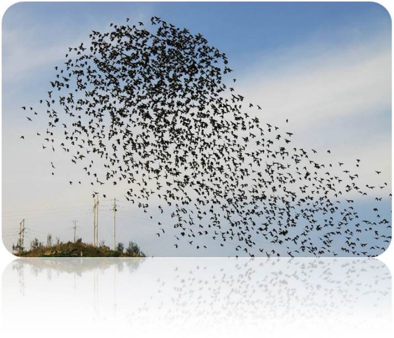
Outline of the presentation

- Introduction
- Problem statement
- Main results
    - State feedback case
    - Output feedback case
- Concluding remarks

Consensus of multi-agent systems

Consensus: Agents achieving an agreement on their common state by using information from the neighbors.



Consensus of multi-agent systems

Theoretical results:

- Finite-time consensus,
- Consensus of high-order systems,
- Consensus with time delay,

⋮

Practical applications:

- Autonomous underwater vehicles,
- Unmanned air vehicles,
- Wireless sensor network,

⋮

Control systems with saturating actuators - fundamental results

$$\dot{x} = Ax + Bu, \quad x \in \mathrm{R}^n, \ ||u||_\infty \leq 1.$$

Null controllable region $\mathcal{C}$:

$$\mathcal{C} = \left\{ x(0) \in \mathrm{R}^n : \ \exists u, \|u\|_\infty \leq 1 \text{ and } T \geq 0, \text{ s.t. } x(T) = 0 \right\}.$$

General characterization of $\mathcal{C}$ [Hsu, PhD Dissertation '76]

Assume that (A, B) is controllable.

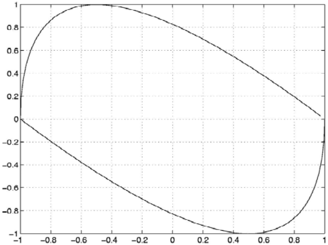
- If A is semi-stable ($\lambda(A) \subset \mathbf{C}^- \cup \mathbf{C}^0$), then, $\mathcal{C} = \mathrm{R}^n$.
- If A is anti-stable ($\lambda(A) \subset \mathbf{C}^+$), then, $\mathcal{C}$ is a bounded convex open set.
- If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $A_1 \in \mathrm{R}^{n_1 \times n_1}$, $A_2 \in \mathrm{R}^{n_2 \times n_2}$, $\lambda(A_1) \subset \mathbf{C}^+$, $\lambda(A_2) \subset \mathbf{C}^- \cup \mathbf{C}^0$, then, $\mathcal{C} = \mathcal{C}_1 \times \mathrm{R}^{n_2}$, where $\mathcal{C}_1$ is the controllable region of $\dot{x}_1 = A_1 x_1 + B_1 \sigma(u)$.

Control systems with saturating actuators - fundamental results

Characterization of $\mathcal{C}_1$

T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhauser, 2001.

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \partial\mathcal{C} = \left\{ \pm \left(-2e^{-At} + I \right) A^{-1} B : \; t \in [0, \infty] \right\}$$



Control systems with saturating actuators - fundamental results

Global and semi-global stabilization

- Global stabilization requires (A, B) to be asymptotically null controllable with bounded controls (ANCBC), *i.e.*, (A, B) is stabilizable and $\lambda(A) \subset \mathbf{C}^- \cup \mathbf{C}^0$ [Sussmann, Sontag & Yang, *CDC* '90]
- in general, nonlinear feedback laws are needed [Fuller, IJC '76; Sussmann & Yang, *CDC* '91]
- nonlinear, but smooth, stabilizers [Sussmann, Sontag & Yang; Teel; Megretski; Lin; $\cdots$, '90s]
- Semi-global stabilization can be achieved with linear feedback [Lin, *Low Gain Feedback*, Springer, '98]

Consensus of multi-agent systems subject to input saturation [Meng, Zhao & Lin, *SCL* '13, Zhao & Lin, *CCC* '13]

Consider a group of N networked follower agents subject to actuator saturation:

$$\dot{x}_i = Ax_i + B\sigma(u_i), \; i = 1, 2, \cdots, N,$$

where $x_i \in \mathrm{R}^n$ is the state, $u_i \in \mathrm{R}^m$ is the input, (A, B) is stabilizable, and $\sigma(u_i) = [\sigma(u_{i1}), \sigma(u_{i2}), \cdots, \sigma(u_{im})]^{\mathrm{T}}$, with $\sigma(u_{ij}) = \mathrm{sign}(u_{ij}) \min\{|u_{ij}|, \Delta\}$.

The trajectory of the leader agent is governed by

$$\dot{x}_0 = Ax_0.$$

Consensus of multi-agent systems subject to input saturation [Meng, Zhao & Lin, *SCL* '13, Zhao& Lin, *CCC* '13]

Global leader-following consensus problem

Design a local feedback law for each follower agent such that

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0,\ i = 1, 2, \cdots, N.$$

Semi-global leader following consensus problem

For any given bounded set $\mathcal{X} \subset \mathrm{R}^n$, design a local feedback law for each follower agent such that, for all $x_i(0) \in \mathcal{X}$, $i = 0, 1, 2, \cdots, N$,

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0,\ i = 1, 2, \cdots, N.$$

Related problems

Consensus problems for multi-agent systems subject to simultaneous actuator position and rate saturation.

Semi-global stabilization of linear systems with position and rate-limited actuators [Lin, *SCL* '97]

Consider a linear system with both position and rate-limited actuators,

$$\begin{cases} \dot{x} = Ax + B\sigma_{\mathrm{p}}(v), \\ \dot{v} = \sigma_{\mathrm{r}}(-v + u), \\ y = Cx, \end{cases}$$

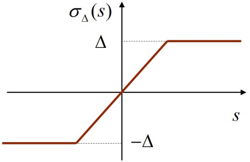
is semi-globally asymptotically stabilizable by linear state feedback if the open loop system is asymptotically null controllable with bounded controls, that is (A, B) is stabilizable in the usual linear systems theory sense and all eigenvalues of A are on the closed left-half plane. If, in addition, (A, C) is detectable, then the system is semi-globally asymptotically stabilizable by linear output feedback.

Problem statement

Consider a group of N networked follower agents with position and rate-limited actuators,

$$\begin{cases} \dot{x}_i = Ax_i + B\sigma_{\mathrm{p}}(v_i), \\ \dot{v}_i = \sigma_{\mathrm{r}}(-v_i + u_i), \quad i = 1, 2, \cdots, N, \end{cases}$$

where saturation functions $\sigma_{\mathrm{p}}(s)$ and $\sigma_{\mathrm{r}}(s)$ are respectively defined as $\sigma_{\mathrm{p}}(s) = \mathrm{sign}(s)\min\{\Delta_{\mathrm{p}}, |s|\}$ and $\sigma_{\mathrm{r}}(s) = \mathrm{sign}(s)\min\{\Delta_{\mathrm{r}}, |s|\}$, for some $\Delta_{\mathrm{p}} > 0$ and $\Delta_{\mathrm{r}} > 0$.



The trajectory of the leader agent is governed by

$$\dot{x}_0 = Ax_0.$$

Problem statement

Semi-global consensus by linear state feedback

For any *a priori* given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, construct a linear state feedback control law u_i for each follower agent, which only uses local information, such that all these feedback control laws together achieve semi-global leader-following consensus, that is, for all $x_i(0) \in \mathcal{X}_0$, $i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0$, $i = 1, 2, \cdots, N$,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, i = 1, 2, \cdots, N.$$

Problem statement

Semi-global consensus by linear output feedback

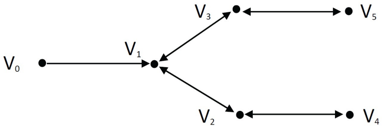
For any *a priori* given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, construct a linear observer based output feedback control law u_i for each follower agent, which only uses local information, such that all these feedback control laws together achieve semi-global leader-following consensus, that is, for all $x_i(0), \hat{x}_i(0) \in \mathcal{X}_0$, $i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0$, $i = 1, 2, \cdots, N$,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, i = 1, 2, \cdots, N,$$

where $\hat{x}$ is the state of the observer.

Graph theory

$$\begin{aligned}
\mathcal{G}_N = \{\mathcal{V}, \mathcal{E}\} &: \text{Graph} \\
\mathcal{V} = \{\nu_1, \nu_2, \cdots, \nu_N\} &: \text{Nodes} \\
\mathcal{E} \in \mathcal{V} \times \mathcal{V} &: \text{Edges} \\
(v_{i1}, v_{i2}), (v_{i2}, v_{i3}), \cdots &: \text{Path} \\
A_N = [a_{ij}] &: \text{Adjacency matrix} \\
L_N = [l_{ij}] &: \text{Laplacian matrix} \\
\mathrm{diag}\{a_{10}, a_{20}, \cdots, a_{N0}\} &: \text{Connection between followers and leader}
\end{aligned}$$



An undirected graph

Preliminaries

Assumption 1

The undirected graph $\mathcal{G}_N$ is connected and $a_{i0} > 0$ for at least one $i, i = 1, 2, \cdots, N$.

Denote $M = L_N + \mathrm{diag}\{a_{10}, a_{20}, \cdots, a_{N0}\}$.

Lemma 1 [Hu & Hong, *Physica A* '07]

Let Assumption 1 hold. Then, M is symmetric and positive definite.

For the positive definite matrix M, we order its eigenvalues as $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Preliminaries

Assumption 2

The pair (A, B) is stabilizable and all eigenvalues of A are on the closed left-half plane.

Lemma 2[Lin, *Low Gain Feedback*, London'98]

Let Assumption 2 hold. Then, for each $\varepsilon \in (0, 1]$, there exists a unique matrix $P(\varepsilon) > 0$ that solves the algebraic Riccati equation (ARE)

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P + \varepsilon I = 0.$$

Moreover, such a $P(\varepsilon)$ satisfies

1. $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$;
2. There exists a constant $\alpha > 0$, independent of ε, such that
$$\left\| P^{\frac{1}{2}}(\varepsilon) A P^{-\frac{1}{2}}(\varepsilon) \right\| \leq \alpha, \ \varepsilon \in (0, 1].$$

Preliminaries

Corollary 1

Let Assumptions 1 and 2 hold. Then, there exists a unique positive definite solution $P(\varepsilon)$ to the following algebraic Riccati equation

$$A^{\mathsf{T}}P + PA - \gamma PBB^{\mathsf{T}}P + \varepsilon I = 0, \quad \varepsilon \in (0, 1]$$

where γ is any positive constant such that $\gamma \leq \lambda_1$.

Lemma 3

Let Assumptions 1 and 2 hold. Then, for any vector $x \in \mathrm{R}^{Nn}$, the following inequality always holds,

$$x^{\mathsf{T}}(M^2 \otimes PBB^{\mathsf{T}}P)x \geq \gamma x^{\mathsf{T}}(M \otimes PBB^{\mathsf{T}}P)x,$$

where $\gamma \leq \lambda_1$.

Semi-global consensus by linear state feedback

Control laws:

$$u_i = -\frac{1}{\varepsilon^2} B^{\mathsf{T}} P(\varepsilon) \left(\sum_{j=1}^{N} a_{ij}(x_i - x_j) + a_{i0}(x_i - x_0) \right) - \left(\frac{1}{\varepsilon^2} - 1 \right) v_i,$$

$$i = 1, 2, \cdots, N.$$

Theorem 1

Let Assumptions 1 and 2 hold. Then, under the linear state feedback control laws, the group of follower agents and the leader agent achieve semi-global leader-following consensus. That is, for any given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, there is an $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

for all $x_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

Proof of Theorem 1

Let the error states be $\bar{x}_i = x_i - x_0, i = 1, 2, \cdots, N$. Denote $\bar{x} = [\bar{x}_1^{\mathrm{T}}, \bar{x}_2^{\mathrm{T}}, \cdots, \bar{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, $v = [v_1^{\mathrm{T}}, v_2^{\mathrm{T}}, \cdots, v_N^{\mathrm{T}}]^{\mathrm{T}}$ and $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \cdots, u_N^{\mathrm{T}}]^{\mathrm{T}}$. Then we have

$$\begin{cases} \dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)\sigma_{\mathrm{p}}(v), \\ \dot{v} = \sigma_{\mathrm{r}}(-v + u). \end{cases}$$

Notice that the state feedback control laws can be written as

$$u = -\frac{1}{\varepsilon^2}\left(v + (M \otimes B^{\mathrm{T}} P)\bar{x}\right) + v.$$

Construct a Lyapunov function,

$$V(\bar{x}, v) = \bar{x}^{\mathrm{T}}(M \otimes P)\bar{x} + \left(v + (M \otimes B^{\mathrm{T}} P)\bar{x}\right)^{\mathrm{T}}\left(v + (M \otimes B^{\mathrm{T}} P)\bar{x}\right),$$

which is positive definite since M and P are both positive definite.

Proof of Theorem 1

Let $c > 0$ be a constant scalar such that

$$c \geq \sup_{\varepsilon \in (0,1], x_i \in \mathcal{X}_0, i=0,1,\cdots,N, v_i \in \mathcal{V}_0, i=1,2,\cdots,N} V(\bar{x}, v).$$

Such a c exists since $\mathcal{X}_0$ and $\mathcal{V}_0$ are both bounded and $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$.

Let $L_V(c) := \left\{(\bar{x}, v) \in \mathbb{R}^{N(n+m)} : V(\bar{x}, v) \leq c\right\}$. Let $\varepsilon_1^* \in (0, 1]$ be such that for all $\varepsilon \in (0, \varepsilon_1^*]$, $(\bar{x}, v) \in L_V(c)$ implies that

$$\begin{aligned} \|(M \otimes B^{\mathrm{T}} P(\varepsilon))\bar{x}\| &\leq \frac{\Delta}{3}, \\ \|(M \otimes B^{\mathrm{T}} P(\varepsilon) A)\bar{x}\| &\leq \frac{\Delta}{3Nm}, \\ \|(M \otimes B^{\mathrm{T}} P(\varepsilon) B)\sigma_{\mathrm{p}}(v)\| &\leq \frac{\Delta}{3Nm}, \end{aligned}$$

where $\Delta = \min\{\Delta_{\mathrm{p}}, \Delta_{\mathrm{r}}\}$.

Proof of Theorem 1

The derivative of V along the trajectories of the closed-loop system inside the level set $L_V(c)$ can be evaluated as follows,

$$\dot{V} \leq -\varepsilon \bar{x}^{\mathsf{T}}(M \otimes I_n)\bar{x} + 2\sum_{k=1}^{Nm}\Bigg(- \mu^k \left(\sigma_{\rm p}\left(v^k\right) - \mu^k\right) + \left(v^k - \mu^k\right) \times \left(\sigma_{\rm r}\left(-\frac{1}{\varepsilon^2}\left(v^k - \mu^k\right)\right) + F^k\bar{x} + K^k\sigma_{\rm p}(v)\right)\Bigg),$$

where v^k is the kth element of $v = [v_1^{\mathsf{T}}, v_2^{\mathsf{T}}, \cdots, v_N^{\mathsf{T}}]^{\mathsf{T}}$, μ^k is the kth element of $\mu = -(M \otimes B^{\mathsf{T}}P)\bar{x}$, F^k is the kth column of matrix $F = (M \otimes B^{\mathsf{T}}PA)$ and K^k is the kth column of matrix $(M \otimes B^{\mathsf{T}}PB)$.

Proof of Theorem 1

Discuss the derivative of V under the following three cases:

- $|v^k - \mu^k| > \varepsilon^2\Delta$ for all $k = 1, 2, \cdots, Nm$,
- $|v^k - \mu^k| < \varepsilon^2\Delta$ for at least one k but not all k, $k = 1, 2, \cdots, Nm$,
- $|v^k - \mu^k| \leq \varepsilon^2\Delta$ for all $k = 1, 2, \cdots, Nm$.

We arrive at the conclusion that, there exists an $\varepsilon^* \in (0, \varepsilon_1^*]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} < 0, \forall(\bar{x}, v) \in L_v(c) \setminus \{0\}.$$

This implies that the closed-loop system is asymptotically stable at $(\bar{x}, v) = (0, 0)$ with $L_V(c)$ included in the domain of attraction, and hence,

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0$, $i = 0, 1, \cdots, N$ and $v_i(0) \in \mathcal{V}_0$, $i = 1, 2, \cdots, N$.

Semi-global consensus by linear output feedback

Assumption 3

The pair (A, C) is detectable.

Construct a state observer for each agent as follows,

$$\dot{\hat{x}}_i = A\hat{x}_i + L(y_i - C\hat{x}_i), \; i = 0, 1, \cdots, N,$$

where $\hat{x}_i \in \mathbb{R}^n$ and L is any matrix such that $A - LC$ is Hurwitz.

Control laws:

$$u_i = -\frac{1}{\varepsilon^2} B^{\mathrm{T}} P(\varepsilon) \left(\sum_{j=1}^{N} a_{ij}(\hat{x}_i - \hat{x}_j) + a_{i0}(\hat{x}_i - \hat{x}_0) \right) - \left(\frac{1}{\varepsilon^2} - 1 \right) v_i, \quad i = 1, 2, \cdots, N.$$

Semi-global consensus by linear output feedback

Theorem 2

Let Assumptions 1, 2 and 3 hold. Then, under the linear output feedback control laws, the group of follower agents and the leader agent achieve semi-global leader-following consensus. That is, for any given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, there is an $\varepsilon^* > 0$ such that, for any given $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

for all $x_i(0), \hat{x}_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

Proof of Theorem 2

Let $e_i = x_i - \hat{x}_i, i = 0, 1, \cdots, N$. We have

$$\begin{cases} \dot{e}_0 = (A - LC)e_0, \\ \dot{e}_i = (A - LC)e_i + B\sigma_{\rm p}(v_i), \ \ i = 1, 2, \cdots, N. \end{cases}$$

Let $\bar{e}_i = e_i - e_0, i = 1, 2, \cdots, N$, then we have

$$\dot{\bar{e}}_i = (A - LC)\bar{e}_i + B\sigma_{\rm p}(v_i), \ \ i = 1, 2, \cdots, N.$$

Denote $\bar{x}_i = x_i - x_0, \bar{y}_i = y_i - y_0, i = 1, 2, \cdots, N$. Then we have

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B\sigma_{\rm p}(v_i), \\ \dot{v}_i = \sigma_{\rm r}(-v_i + u_i), \\ \dot{\bar{e}}_i = (A - LC)\bar{e}_i + B\sigma_{\rm p}(v_i), \\ u_i = -\frac{1}{\varepsilon^2}B^{\rm T}P\Big(\sum_{j=1}^N a_{ij}((\bar{x}_i - \bar{x}_j) - (\bar{e}_i - \bar{e}_j)) + a_{i0}(\bar{x}_i - \bar{e}_i)\Big) - \big(\frac{1}{\varepsilon^2} - 1\big)v_i, \\ \qquad\qquad i = 1, 2, \cdots, N. \end{cases}$$

Proof of Theorem 2

Let $\bar{x} = [\bar{x}_1^\mathrm{T}, \bar{x}_2^\mathrm{T}, \cdots, \bar{x}_N^\mathrm{T}]^\mathrm{T}$, $v = [v_1^\mathrm{T}, v_2^\mathrm{T}, \cdots, v_N^\mathrm{T}]^\mathrm{T}$, $u = [u_1^\mathrm{T}, u_2^\mathrm{T}, \cdots, u_N^\mathrm{T}]^\mathrm{T}$, and $\bar{e} = [\bar{e}_1^\mathrm{T}, \bar{e}_2^\mathrm{T}, \cdots, \bar{e}_N^\mathrm{T}]^\mathrm{T}$. Then we have

$$\begin{cases} \dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)\sigma_\mathrm{p}(v), \\ \dot{v} = \sigma_\mathrm{r}(-v + u), \\ \dot{\bar{e}} = (I_N \otimes (A - LC))\bar{e} + (I_N \otimes B)\sigma_\mathrm{p}(v), \\ u = -\frac{1}{\varepsilon^2}\left(M \otimes B^\mathrm{T} P\right)(\bar{x} - \bar{e}) - \left(\frac{1}{\varepsilon^2} - 1\right) v. \end{cases}$$

Construct a Lyapunov function,

$$\begin{aligned} V(\bar{x}, v, \bar{e}) = {} & \bar{x}^\mathrm{T}(M \otimes P)\bar{x} + \lambda_{\max}^{1/2}(P)\bar{e}^\mathrm{T}(M \otimes P_0)\bar{e} + \left(v + (M \otimes B^\mathrm{T} P)\,\bar{x}\right. \\ & \left. - (M \otimes B^\mathrm{T} P)\,\bar{e}\right)^\mathrm{T} \left(v + (M \otimes B^\mathrm{T} P)\,\bar{x} - (M \otimes B^\mathrm{T} P)\,\bar{e}\right), \end{aligned}$$

where $P_0 > 0$ is the unique positive definite solution to the following Lyapunov equation,

$$(A - LC)^\mathrm{T} P_0 + P_0(A - LC) = -I.$$

Proof of Theorem 2

Let $c > 0$ be a constant scalar such that

$$c \geq \sup_{\varepsilon \in (0,1], x_i, \hat{x}_i \in \mathcal{X}_0, i=0,1,\cdots,N, v_i \in \mathcal{V}_0, i=1,2,\cdots,N} V(\bar{x}, v, \bar{e}).$$

Such a c exists since $\mathcal{X}_0$ and $\mathcal{V}_0$ are both bounded and $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$.

Let $L_V(c) := \left\{(\bar{x}, v, \bar{e}) \in \mathbb{R}^{N(2n+m)} : V(\bar{x}, v, \bar{e}) \leq c\right\}$. Let $\varepsilon_1^* \in (0,1]$ be such that, for all $\varepsilon \in (0, \varepsilon_1^*]$, $(\bar{x}, v, \bar{e}) \in L_V(c)$ implies that,

$$\|(M \otimes B^{\mathrm{T}} P)\,\bar{x}\| \leq \frac{\Delta}{8}, \quad \|(M \otimes B^{\mathrm{T}} P)\,\bar{e}\| \leq \frac{\Delta}{8}, \quad \|(M \otimes B^{\mathrm{T}} PA)\,\bar{x}\| \leq \frac{\Delta}{8Nm},$$

$$\left\|\lambda_{\max}^{1/2}(P)\,(M \otimes B^{\mathrm{T}} P_0)\,\bar{e}\right\| \leq \frac{\Delta}{8}, \quad \|(M \otimes B^{\mathrm{T}} P(A - LC))\,\bar{e}\| \leq \frac{\Delta}{8Nm},$$

where $\Delta = \min\{\Delta_{\mathrm{p}}, \Delta_{\mathrm{r}}\}$.

Proof of Theorem 2

The derivative of V inside the level set $L_V(c)$ can be evaluated as follows,

$$\dot{V} \leq -\varepsilon \bar{x}^{\mathsf{T}}(M \otimes I_n)\bar{x} - \lambda_{\max}^{1/2}(P)\bar{e}^{\mathsf{T}}(M \otimes I_n)\bar{e} + 2\sum_{k=1}^{Nm}\Bigg(-\lambda_{\max}^{1/2}(P)\eta^k \sigma_{\mathrm{p}}\left(v^k\right)$$

$$-\frac{1}{4}\left(\phi^k\right)^2\Bigg) + 2\sum_{k=1}^{Nm}\Bigg(-\frac{1}{4}\left(\phi^k\right)^2 - \phi^k\left(\sigma_{\mathrm{p}}\left(v^k\right) - \phi^k\right)$$

$$+\left(v^k - \phi^k + \omega^k\right)\left(\sigma_{\mathrm{r}}\left(-\frac{1}{\varepsilon^2}\left(v^k - \phi^k + \omega^k\right)\right) + G^k\bar{x} + H^k\bar{e}\right)\Bigg),$$

where v^k is the kth element of v, η^k is the kth element of $\eta = -\left(M \otimes B^{\mathsf{T}}P_0\right)\bar{e}$, ϕ^k is the kth element of $\phi = -\left(M \otimes B^{\mathsf{T}}P\right)\bar{x}$, ω^k is the kth element of $\omega = -\left(M \otimes B^{\mathsf{T}}P\right)\bar{e}$, G^k is the kth column of matrix $G = M \otimes B^{\mathsf{T}}PA$, and H^k is the kth column of matrix $H = M \otimes B^{\mathsf{T}}P(A - LC)$.

Proof of Theorem 2

Discuss the derivative of V under the following three cases:

- $\left|v^k - \phi^k + \omega^k\right| > \varepsilon^2\Delta$ for all $k = 1, 2, \cdots, Nm$,
- $|\left|v^k - \phi^k + \omega^k\right| > \varepsilon^2\Delta$ for at least one k but not all $k, k = 1, 2, \cdots, Nm$,
- $\left|v^k - \phi^k + \omega^k\right| \leq \varepsilon^2\Delta$ for all $k = 1, 2, \cdots, Nm$.

We can conclude that, there exists an $\varepsilon \in (0, \varepsilon_1^*]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} < 0, \forall(\bar{x}, v) \in L_v(c) \setminus \{0\}.$$

This implies that the closed-loop system is asymptotically stable at $(\bar{x}, v) = (0, 0)$ with $L_V(c)$ included in the domain of attraction, and hence,

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

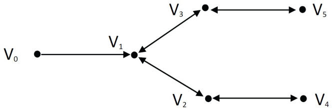
An example

Consider a group of 5 agents and a leader with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and $\Delta_{\rm p} = 5, \Delta_{\rm r} = 0.5$.

The communication topology among agents is as shown below:



An example

For the given graph, we have

$$M = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The minimum eigenvalue of M is $\lambda_{\mathrm{min}}(M) = 0.1392$.

We choose $\gamma = 0.01 < \lambda_{\mathrm{min}}(M)$.

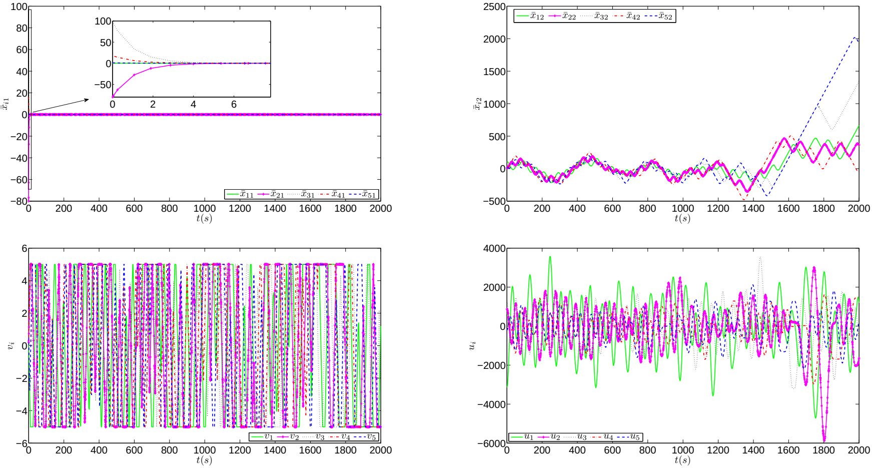
Simulation results - state feedback

Choose the initial values of the agents randomly as

$$\begin{aligned} & \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) & x_5(0) & x_0(0) \end{bmatrix} \\ = & \begin{bmatrix} -10 & 0.1 & -80 & 98 & 18 & 1 \\ 10 & 108 & 10 & -0.1 & -0.5 & 20 \end{bmatrix}, \\ & \begin{bmatrix} v_1(0) & v_2(0) & v_3(0) & v_4(0) & v_5(0) \end{bmatrix} \\ = & \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}. \end{aligned}$$

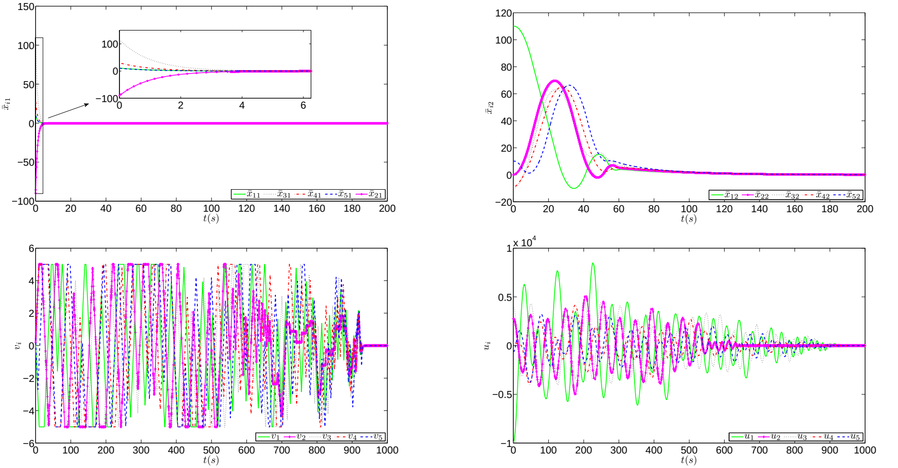
We will simulate the closed-loop system for two different values of ε, $\varepsilon = 1$ and $\varepsilon = 0.1$.

Simulation results - state feedback



Evolutions of the agents under state feedback control laws when $\varepsilon = 1$.

Simulation results - state feedback



Evolutions of the agents under state feedback control laws when $\varepsilon = 0.1$.

Simulation results - output feedback

Choose the initial values of the agents randomly as

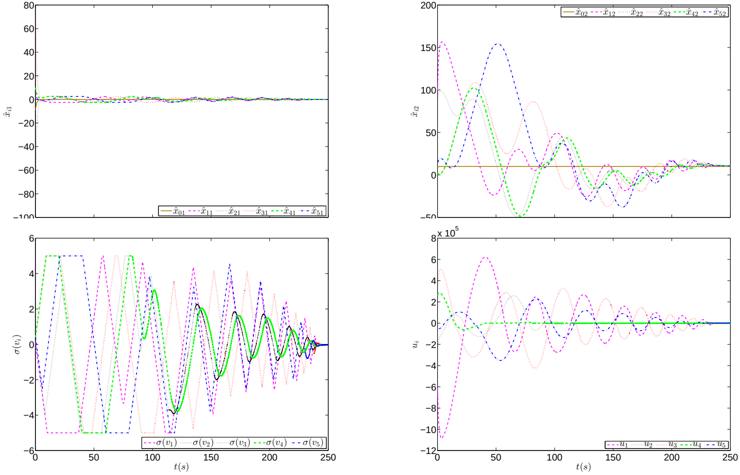
$$
\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) & x_5(0) & x_0(0) \end{bmatrix}
= \begin{bmatrix} -10 & 0.1 & -80 & 98 & 18 & 1 \\ 10 & 108 & 10 & -0.1 & -0.5 & 20 \end{bmatrix},
$$

$$
\begin{bmatrix} v_1(0) & v_2(0) & v_3(0) & v_4(0) & v_5(0) \end{bmatrix}
= \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}.
$$

Choose the initial values of observer of the agents randomly as

$$
\begin{bmatrix} \hat{x}_0(0) & \hat{x}_1(0) & \hat{x}_2(0) & \hat{x}_3(0) & \hat{x}_4(0) & \hat{x}_5(0) \end{bmatrix}
= \begin{bmatrix} -9 & 10 & -90 & 70 & 10 & -1 \\ 9 & 100 & 90 & 1 & 2 & 13 \end{bmatrix}.
$$

We will simulate the closed-loop system $\varepsilon = 0.01$.



Evolutions of the agents under output feedback control laws with $\varepsilon = 0.01$.

Conclusions

We studied the semi-global leader-following consensus problem for a group of linear systems in the presence of both actuator position and rate saturation.

We constructed both a family of linear state feedback control laws and a family of linear output feedback control laws for each follower agent by using low gain feedback design strategy, which only uses the information of agent and its neighbors.

Semi-global leader-following consensus can be achieved by using the proposed control laws when the communication topology among follower agents is a connected undirected graph and the leader is a neighbor of at least one follower.

Great challenges remain when the agents are open loop exponentially unstable.

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